Domino Tilings of an $m \times n$ Chess Board

Dhruv Ajmera

The University of Texas at Austin

December 2025

What are we trying to solve?

First, we provide some definitions for the sake of clarity.

What are we trying to solve?

First, we provide some definitions for the sake of clarity.

- We define a Chessboard as an $m \times n$ rectangular grid.

What are we trying to solve?

First, we provide some definitions for the sake of clarity.

- We define a Chessboard as an $m \times n$ rectangular grid.
- We define a Domino as a 2×1 rectangle.

What are we trying to solve?

First, we provide some definitions for the sake of clarity.

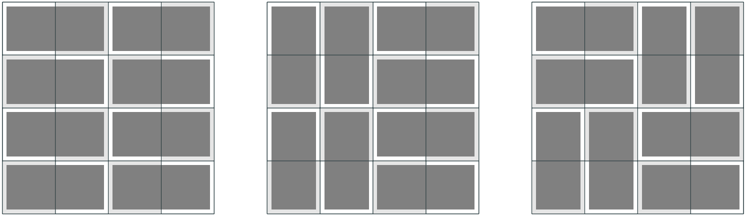
- We define a Chessboard as an $m \times n$ rectangular grid.
- We define a Domino as a 2×1 rectangle.
- We define a Tiling of a Chessboard as a perfect covering of this grid with Dominoes such that the Dominoes entirely cover the Board with no overlaps, protrusions or diagonal placements.

An Example

Below, we show some examples of tilings of a 4×4 chessboard.

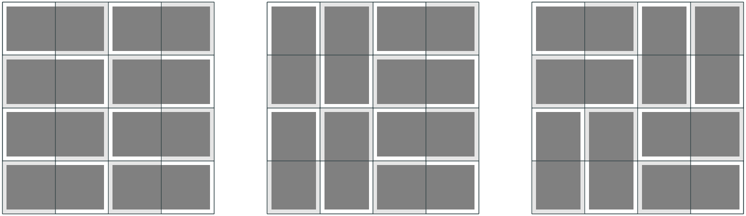
An Example

Below, we show some examples of tilings of a 4×4 chessboard.



An Example

Below, we show some examples of tilings of a 4×4 chessboard.



But how many tilings are there for a general m, n?

Some Basic Facts

To help build our intuition for tilings, we provide a basic fact/requirement.

Some Basic Facts

To help build our intuition for tilings, we provide a basic fact/requirement.

- In order to tile a Chessboard, it is necessary that mn is even (i.e. both m, n cannot be odd).

Proving Existence of Tilings

The existence of tilings requires that mn is even.

Proving Existence of Tilings

The existence of tilings requires that mn is even.

Proof.

If mn is odd, this means that there are an odd number of squares on our chessboard. Dominoes can only cover an even number of squares since each domino is 2×1. Thus, it is impossible for dominoes to tile a chessboard with an odd number of squares. $\square$

Proving Existence of Tilings

The existence of tilings requires that mn is even.

Proof.
If mn is odd, this means that there are an odd number of squares on our chessboard. Dominoes can only cover an even number of squares since each domino is 2×1. Thus, it is impossible for dominoes to tile a chessboard with an odd number of squares. $\square$

If mn is even, we can construct a tiling as follows. WLOG, assume m is even. We place $\frac{m}{2}$ Dominoes vertically in each column. This tiles the Chessboard.

A Recursive Solution

If we have a $2 \times n$ chessboard, we can express our solution recursively:

A Recursive Solution

If we have a $2 \times n$ chessboard, we can express our solution recursively:

$$T(2, n) = T(2, n-1) + T(2, n-2)$$

A Recursive Solution

If we have a $2 \times n$ chessboard, we can express our solution recursively:

$$T(2,n) = T(2,n-1) + T(2,n-2)$$

This follows from the fact that a tiling of a $2 \times n$ chessboard ends with either one vertical domino or two horizontal dominoes. Although this can be extended further, it quickly becomes too complicated for our general solution.

The Final Equation

In fact, we can derive a closed form equation for our general case!

The Final Equation

In fact, we can derive a closed form equation for our general case!

$$T(m,n) = \left(\prod_{j=1}^{m}\prod_{k=1}^{n}\left(4\cos^2\frac{\pi j}{m+1} + 4\cos^2\frac{\pi k}{n+1}\right)\right)^{\frac{1}{4}}$$

The Final Equation

In fact, we can derive a closed form equation for our general case!

$$T(m,n) = \left(\prod_{j=1}^{m}\prod_{k=1}^{n}\left(4\cos^2\frac{\pi j}{m+1} + 4\cos^2\frac{\pi k}{n+1}\right)\right)^{\frac{1}{4}}$$

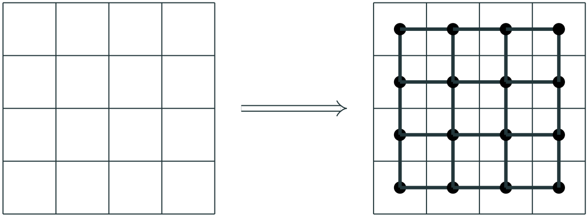
Despite seeming like this shouldn't even be an integer, it actually counts the number of tilings! However, to derive this, we must change our perspective.

Our Strategy

We will reformulate this problem into a graph theory and linear algebra problem to solve it!

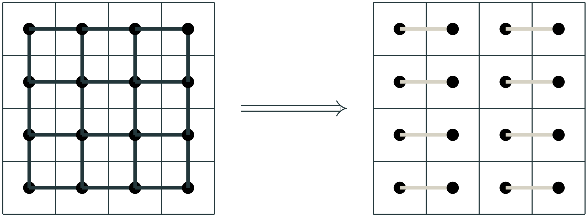
The Setup

- We represent the Chessboard with a vertex in each square and an edge connecting all adjacent vertices (nondiagonally).



The Setup

- We represent the Chessboard with a vertex in each square and an edge connecting all adjacent vertices (nondiagonally).
- A Tiling corresponds to a selection of these edges that perfectly covers every vertex once (only one edge should be leaving each vertex). This is also called a perfect matching.



Formal Definitions

Our graph is both bipartite and a perfect matching. Formally,

Formal Definitions

Our graph is both bipartite and a perfect matching. Formally,

1. A graph is bipartite if its vertex set V can be partitioned into two sets $Y, Z \subset V$ such that these sets share no common elements ($Y \cap Z = \emptyset$), both these sets together are the vertex set ($Y \cup Z = V$) and each vertex in one set is connected to exactly one vertex in the other.

Formal Definitions

Our graph is both bipartite and a perfect matching. Formally,

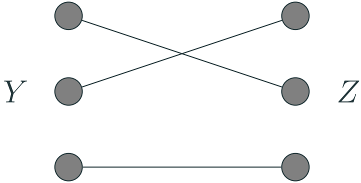
1. A graph is bipartite if its vertex set V can be partitioned into two sets $Y, Z \subset V$ such that these sets share no common elements ($Y \cap Z = \emptyset$), both these sets together are the vertex set ($Y \cup Z = V$) and each vertex in one set is connected to exactly one vertex in the other.

2. A graph is a perfect matching when each vertex is connected to exactly one other vertex. (Each vertex must only have one edge originating from itself).

Below is a visual representation of what it means to be bipartite.

Bipartite

Below is a visual representation of what it means to be bipartite.

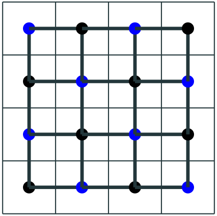


Bipartiteness of the Graph

We provide the following partition of vertices as proof that our graph is bipartite.

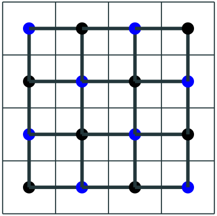
Bipartiteness of the Graph

We provide the following partition of vertices as proof that our graph is bipartite.



Bipartiteness of the Graph

We provide the following partition of vertices as proof that our graph is bipartite.



Here, the blue vertices are the set Y and the black vertices are the set Z. Clearly, they fulfill the promised properties, making our graph bipartite.

Setting up the Derivation

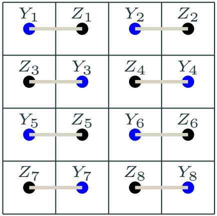
Given dimensions m, n, our grid graph has mn vertices. Thus, each partition Y, Z has $N := \frac{mn}{2}$ vertices.

Setting up the Derivation

Given dimensions m, n, our grid graph has mn vertices. Thus, each partition Y, Z has $N := \frac{mn}{2}$ vertices. In each Y, Z, number the vertices $Y_1, \ldots, Y_N$ and $Z_1, \ldots, Z_N$. The method of ordering is arbitrary, but for the sake of simplicity, we will go left to right for all rows.

Setting up the Derivation

Given dimensions m, n, our grid graph has mn vertices. Thus, each partition Y, Z has $N := \frac{mn}{2}$ vertices. In each Y, Z, number the vertices $Y_1, \ldots, Y_N$ and $Z_1, \ldots, Z_N$. The method of ordering is arbitrary, but for the sake of simplicity, we will go left to right for all rows.

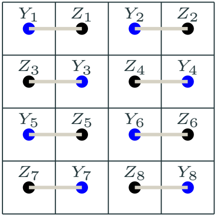


Significance

With this new representation, we can utilize the properties of graphs to our advantage. Namely, we will make use of the adjacency matrix of the graph G.

Significance

With this new representation, we can utilize the properties of graphs to our advantage. Namely, we will make use of the adjacency matrix of the graph G.



$$\Longrightarrow \begin{array}{cc} & Y_i \\ Z_i & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Perfect matching on a bipartite graph and its adjacency matrix.

Significance

With this new representation, we can utilize the properties of graphs to our advantage. Namely, we will make use of the adjacency matrix of the graph G.

Definition 1

The Adjacency Matrix of a graph of n vertices is an $n \times n$ matrix $A = [a_{i,j}]$, where

- $a_{i,j} = 1$ if there is an edge from vertex v_i to v_j,
- $a_{i,j} = 0$ otherwise.

Specifically, since our graph is undirected, the matrix is symmetric $(a_{i,j} = a_{j,i})$.

In our case, our adjacency matrix is that of a bipartite graph, so it has dimensions $N \times N$, $N := \frac{mn}{2}$.

Permutations

Definition 2

A Permutation $w \in \mathscr{G}_N$ is a bijective function $w : \{1, \dots, N\} \to \{1, \dots, N\}$. We write this permutation as a 'word' $w = w(1)w(2)\dots w(N)$.

Permutations

Definition 3

A Permutation $w \in \mathscr{G}_N$ is a bijective function $w : \{1, \ldots, N\} \to \{1, \ldots, N\}$. We write this permutation as a 'word' $w = w(1)w(2)\ldots w(N)$.

An example of a permutation could be $w(x) = x + 1 \pmod 8$, $x \in \{1, \ldots, 8\}$. This would yield $w = w(1)w(2)\ldots w(N)$, $w = 23456781$.

Permutations

Definition 4

A Permutation $w \in \mathscr{G}_N$ is a bijective function $w : \{1, \ldots, N\} \to \{1, \ldots, N\}$. We write this permutation as a 'word' $w = w(1)w(2)\ldots w(N)$.

An example of a permutation could be $w(x) = x + 1 \pmod 8$, $x \in \{1, \ldots, 8\}$. This would yield $w = w(1)w(2)\ldots w(N)$, $w = 23456781$.

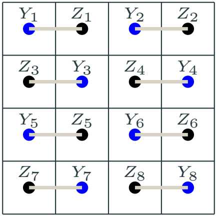
We will utilize permutations as a way to represent perfect matchings, aiding in our calculation.

Connection to Permutations

Now, since Y, Z each have N labeled vertices, any given perfect matching can be completely represented by a permutation $w \in \mathscr{G}_N$. Given $w(i) = j$, this corresponds to the vertex $y_i \in Y$ being adjacent to $z_j \in Z$.

Connection to Permutations

Now, since Y, Z each have N labeled vertices, any given perfect matching can be completely represented by a permutation $w \in \mathscr{G}_N$. Given $w(i) = j$, this corresponds to the vertex $y_i \in Y$ being adjacent to $z_j \in Z$.



For example, this graph corresponds to the identity permutation $(12345678 \rightarrow 12345678)$.

However, it is important to note that an arbitrary permutation w does not necessarily represent a perfect matching in G (since the selected vertices may not be adjacent).

Connection to Permutations (cont.)

However, it is important to note that an arbitrary permutation w does not necessarily represent a perfect matching in G (since the selected vertices may not be adjacent).

For example, the permutation $12345678 \to 87654321$ does not represent a perfect matching, since no edge can exist between Y_1 and Z_8 (among other discrepancies).

Connection to Permutations (cont.)

However, it is important to note that an arbitrary permutation w does not necessarily represent a perfect matching in G (since the selected vertices may not be adjacent).

For example, the permutation $12345678 \rightarrow 87654321$ does not represent a perfect matching, since no edge can exist between Y_1 and Z_8 (among other discrepancies).

With this constraint, we ask: how do we know if a permutation does represent a perfect matching?

Counting with Permutations

w represents a perfect matching iff all of the edges represented by w are present in G.

Counting with Permutations

w represents a perfect matching iff all of the edges represented by w are present in G.

More precisely, this means that we require all $a_{i,w(i)} \neq 0$. This is equivalent to requiring

$$\prod a_{i,w(i)} = 1.$$

Counting Tilings via the Adjacency Matrix

As a result,

$$T(m,n) = \sum_{w \in \mathscr{G}_N} \prod_{i=1}^{N} a_{i,w(i)}.$$

(A permutation that represents a matching in G adds 1 to the sum, whereas non-representative permutations add 0).

Connecting to Known Formulas

Definition 5

Given an $N \times N$ matrix A, the permanent of A is

$$\operatorname{perm}(A) = \sum_{w \in \mathscr{G}_N} \prod_{i=1}^{N} a_{i,w(i)}$$

and the determinant of A is

$$\det(A) = \sum_{w \in \mathscr{G}_N} \operatorname{sgn}(w) \prod_{i=1}^{N} a_{i,w(i)}.$$

Here, $\operatorname{sgn}(w) = \pm 1$, dependent on the parity of w (the number of transpositions required to return w to the identity).

The Problem

Although the permanent of our matrix is equivalent to the number of tilings, calculating the permanent of any matrix is a computationally expensive task. Thus, we seek to find another matrix $\bar{A}$ such that we can relate the determinant of $\bar{A}$ to the permanent of A.

To compute perfect matchings efficiently, we will create the $N \times N$ Kasteleyn Matrix $K = [k_{i,j}]$ by assigning weights/signs to edges to introduce strategic cancellations in the determinant:

The Kasteleyn Matrix

To compute perfect matchings efficiently, we will create the $N \times N$ Kasteleyn Matrix $K = [k_{i,j}]$ by assigning weights/signs to edges to introduce strategic cancellations in the determinant:

$$k_{i,j} = \begin{cases} w_{i,j} & \text{if vertex } y_i \text{ is adjacent to vertex } z_j, \\ 0 & \text{otherwise.} \end{cases}$$

The Kasteleyn Matrix

To compute perfect matchings efficiently, we will create the $N \times N$ Kasteleyn Matrix $K = [k_{i,j}]$ by assigning weights/signs to edges to introduce strategic cancellations in the determinant:

$$k_{i,j} = \begin{cases} w_{i,j} & \text{if vertex } y_i \text{ is adjacent to vertex } z_j, \\ 0 & \text{otherwise.} \end{cases}$$

For our specific case, the canonical weighting is

- $w_{i,j} = 1$ for horizontal edges,
- $w_{i,j} = i$ (yes, the imaginary unit!) for vertical edges.

The Kasteleyn Matrix

To compute perfect matchings efficiently, we will create the $N \times N$ Kasteleyn Matrix $K = [k_{i,j}]$ by assigning weights/signs to edges to introduce strategic cancellations in the determinant:

$$k_{i,j} = \begin{cases} w_{i,j} & \text{if vertex } y_i \text{ is adjacent to vertex } z_j, \\ 0 & \text{otherwise.} \end{cases}$$

For our specific case, the canonical weighting is

- $w_{i,j} = 1$ for horizontal edges,
- $w_{i,j} = i$ (yes, the imaginary unit!) for vertical edges.

This weighting is chosen to cancel out the sign of the permutation when calculating the determinant of the matrix.

Kasteleyn's Theorem

Specifically, We will relate this to Kasteleyn's Theorem.

Kasteleyn's Theorem

Specifically, We will relate this to Kasteleyn's Theorem.

Theorem 7

Define the Kasteleyn matrix of G to be the $V \times V$ matrix:

$$k_{i,j} = \begin{cases} 1 & (i,j) \text{ is a horizontal edge} \\ i & (i,j) \text{ is a vertical edge} \\ 0 & \text{otherwise} \end{cases}$$

then

$$\#\{\text{Perfect Matchings of } G\} = \sqrt{|det(K)|}$$

Kasteleyn's Theorem

Specifically, We will relate this to Kasteleyn's Theorem.

Theorem 8

Define the Kasteleyn matrix of G to be the $V \times V$ matrix:

$$k_{i,j} = \begin{cases} 1 & (i,j) \text{ is a horizontal edge} \\ i & (i,j) \text{ is a vertical edge} \\ 0 & \text{otherwise} \end{cases}$$

then

$$\#\{\text{Perfect Matchings of } G\} = \sqrt{|det(K)|}$$

Wait, but where did the square root come from?

Bridging the Gap

Kasteleyn's Theorem takes into account the entire vertex set of the graph.

Bridging the Gap

Kasteleyn's Theorem takes into account the entire vertex set of the graph. Denote the $V \times V$ Kasteleyn Matrix as K and the $\frac{mn}{2} \times \frac{mn}{2}$ Kasteleyn Matrix as K^*. Then, by this design,

$$K = \begin{pmatrix} 0 & K^* \\ (K^*)^T & 0 \end{pmatrix},$$

Where K is an off-diagonal block matrix.

Bridging the Gap

Kasteleyn's Theorem takes into account the entire vertex set of the graph. Denote the $V \times V$ Kasteleyn Matrix as K and the $\frac{mn}{2} \times \frac{mn}{2}$ Kasteleyn Matrix as K^*. Then, by this design,

$$K = \begin{pmatrix} 0 & K^* \\ (K^*)^T & 0 \end{pmatrix},$$

Where K is an off-diagonal block matrix.

By the formula of the determinant of a block matrix,

$$|\det K^*| = \sqrt{|\det K|}.$$

This accounts for the square root. Now, we can begin solving for the closed-form equation.

Path Graphs

This is a path graph (an edge from each v_i to v_{i+1}):

Path Graphs

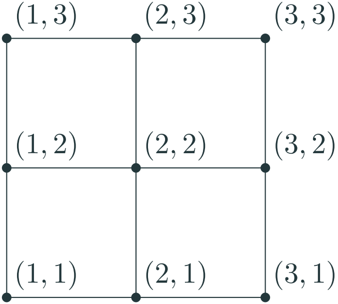
This is a path graph (an edge from each v_i to v_{i+1}):

The Cartesian Product of two such path graphs P_m, P_n is defined as follows: $P_m \times P_n$ has vertices (i, j), where $1 \leq i \leq m, 1 \leq j \leq n$, and two grid points are connected if they are neighbors in one direction and the same in the other.

Thus, it follows that the graph G can be expressed as $G = P_m \times P_n$, where P_m, P_n are path graphs.

The Graph G

Below, we show an example for $m, n = 3$:



Based on this property, one can derive that the Kasteleyn matrix K can be written as

$$K = A_m \otimes I_n + i(I_m \otimes A_n),$$

where the symbol $\otimes$ represents the tensor product of matrices, A_m, A_n are the adjacency matrices of P_m, P_n respectively and I_m, I_n are the identity matrices.

Eigenvalues

By solving $A_i v = \lambda v$ for some nonzero vector v, the eigenvalues of A_n, A_m are

$$\lambda_j = 2\cos\frac{\pi j}{m+1},\ j = 1, 2, \ldots, m,$$

$$\mu_k = 2\cos\frac{\pi k}{n+1},\ k = 1, 2, \ldots, n,$$

respectively.

Eigenvalues

By solving $A_i v = \lambda v$ for some nonzero vector v, the eigenvalues of A_n, A_m are

$$\lambda_j = 2\cos\frac{\pi j}{m+1},\ j = 1, 2, \ldots, m,$$

$$\mu_k = 2\cos\frac{\pi k}{n+1},\ k = 1, 2, \ldots, n,$$

respectively.

Now, we will derive the eigenvalues of K. WLOG, let A_m have an eigenpair (u, λ) and A_n have an eigenpair (v, μ). Define the candidate eigenvector $\psi = u \otimes v$.

Eigenvalues (cont.)

Now, we compute.

$$K\psi = (A_m \otimes I_n)(u \otimes v) + i(I_m \otimes A_n)(u \otimes v),$$

$$K\psi = (A_m u) \otimes (I_n v) + i(I_m u) \otimes (A_n v),$$

$$K\psi = (\lambda u) \otimes v + u \otimes i(\mu v) = \lambda(u \otimes v) + i\mu(u \otimes v),$$

$$K\psi = (\lambda + i\mu)\psi.$$

Eigenvalues (cont.)

Now, we compute.

$$K\psi = (A_m \otimes I_n)(u \otimes v) + i(I_m \otimes A_n)(u \otimes v),$$

$$K\psi = (A_m u) \otimes (I_n v) + i(I_m u) \otimes (A_n v),$$

$$K\psi = (\lambda u) \otimes v + u \otimes i(\mu v) = \lambda(u \otimes v) + i\mu(u \otimes v),$$

$$K\psi = (\lambda + i\mu)\psi.$$

Thus, the eigenvalues of K are

$$\xi_{i,j} = \lambda_j + i\mu_k$$

for all $j = 1, \ldots, m,\ k = 1, \ldots, n.$

Putting it All Together

Now, the determinant of K is the product of all possible eigenvalues (this is a special property of the determinant, and the permanent cannot be computed in this manner):

$$\det(K) = \prod_{i,j} \xi_{i,j} = \prod_{j=1}^{m} \prod_{k=1}^{n} \lambda_j + i\mu_k,$$

$$\det(K) = \prod_{j=1}^{m} \prod_{k=1}^{n} 2\cos\frac{\pi j}{m+1} + 2i\cos\frac{\pi k}{n+1}.$$

Putting it All Together (cont.)

Finally, $\text{perm}(A) = \#\{\text{Perfect Matchings of } G\} = \sqrt{|det(K)|}$.
Thus, we can compute the closed form equation.

Putting it All Together (cont.)

Finally, $\mathrm{perm}(A) = \#\{\text{Perfect Matchings of } G\} = \sqrt{|det(K)|}$.
Thus, we can compute the closed form equation.

$$\sqrt{|\det(K)|} = \left(\prod_{j=1}^{m}\prod_{k=1}^{n}\left|2\cos\frac{\pi j}{m+1} + 2i\cos\frac{\pi k}{n+1}\right|\right)^{\frac{1}{2}}$$

$$\boxed{\#\text{ of tilings} = \left(\prod_{j=1}^{m}\prod_{k=1}^{n}\left(4\cos^2\frac{\pi j}{m+1} + 4\cos^2\frac{\pi k}{n+1}\right)\right)^{\frac{1}{4}}}$$